Duality
Duality

From Wikipedia, Duality may refer to:

In philosophy, logic, and psychology:
- Dualism, a twofold division in several spiritual, religious, and philosophical doctrines
- Dualism (philosophy of mind), where the body and mind are considered to be irreducibly distinct
- De Morgan's Laws, specifically the ability to generate the dual of any logical expression.

In mathematics:
- Duality (mathematics), a mathematical concept
  - Dual (category theory), a formalization of mathematical duality
  - Duality (projective geometry), general principle of projective geometry
  - S-duality (homotopy theory)
Duality

In science:

- Wave-particle duality, a concept in quantum mechanics
- Duality (electrical circuits), regarding isomorphism of electrical circuits
- Duality (mechanical engineering), regarding isomorphism of some mechanical laws
- Duality (electricity and magnetism), regarding isomorphism of physical laws

Physical dualities;

- S-duality
- T-duality
- U-duality
Duality

We encounter many things appear in pairs which compete, conflict, contradict, coordinate, or cooperate each other. For example,

<table>
<thead>
<tr>
<th>Light</th>
<th>Darkness</th>
<th>Male</th>
<th>Female</th>
<th>Voltage</th>
<th>Current</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yang (양)</td>
<td>Yin (음)</td>
<td>Husband</td>
<td>Wife</td>
<td>Controllability</td>
<td>Observability</td>
</tr>
<tr>
<td>Angel</td>
<td>Devil</td>
<td>King</td>
<td>Queen</td>
<td>Potential Energy</td>
<td>Kinetic Energy</td>
</tr>
<tr>
<td>Heaven</td>
<td>Hell</td>
<td>Father</td>
<td>Mother</td>
<td>Electricity</td>
<td>Magnetism</td>
</tr>
<tr>
<td>Soul</td>
<td>Body</td>
<td>Summer</td>
<td>Winter</td>
<td>Wave</td>
<td>Particle</td>
</tr>
<tr>
<td>Good</td>
<td>Evil</td>
<td>Hot</td>
<td>Cold</td>
<td>Day</td>
<td>Night</td>
</tr>
<tr>
<td>God</td>
<td>Satan</td>
<td>Up</td>
<td>Down</td>
<td>Software</td>
<td>Hardware</td>
</tr>
<tr>
<td>Sky</td>
<td>Earth</td>
<td>Positive</td>
<td>Negative</td>
<td>Capitalism</td>
<td>Communism</td>
</tr>
<tr>
<td>Paradise</td>
<td>World</td>
<td>Friend</td>
<td>Enemy</td>
<td>Right Wing</td>
<td>Left Wing</td>
</tr>
<tr>
<td>Truth</td>
<td>Grace</td>
<td>Teaching</td>
<td>Enlightenment</td>
<td>Right Brain</td>
<td>Left Brain</td>
</tr>
<tr>
<td>Peace</td>
<td>War</td>
<td>Law</td>
<td>Love</td>
<td>Faith</td>
<td>Action</td>
</tr>
<tr>
<td>Right</td>
<td>Left</td>
<td>Logos</td>
<td>Pathos</td>
<td>Reason</td>
<td>Emotion</td>
</tr>
</tbody>
</table>
Duality: Electrical Circuits

Duality (electrical circuits)
In electrical engineering, electrical terms are associated into pairs called duals. A dual of a relationship is formed by interchanging voltage and current in an expression. The dual expression thus produced is of the same form, and the reason that the dual is always a valid statement can be traced to the duality of electricity and magnetism.

Here is a partial list of electrical dualities:

- voltage — current,
- resistance — conductance,
- capacitance — inductance,
- short circuit — open circuit,
- two resistances in series — two conductances in parallel;
- Kirchhoff's current law — Kirchhoff's voltage law.
- Thévenin's theorem — Norton's theorem.
Duality in Electric Circuits

We shall define **duality** in terms of the circuit equations. Two circuits are **duals** if the mesh equations that characterize one of them have the same mathematical form as the nodal equations that characterize the other. They are said to be exact duals if each mesh equation of the one circuit is numerically identical with the corresponding nodal equation of the other; the current and voltage variables themselves cannot be identical, of course. Duality itself merely refers to any of the properties exhibited by dual circuits.
Duality Theory

Dual Relationship of Two RLC Circuits

Single Node-pair: Use KCL

\[ -i_S + i_R + i_L + i_C = 0 \]

\[ i_R = \frac{v(t)}{R}; \quad i_L = \frac{1}{L} \int_{t_0}^{t} v(x)dx + i_L(t_0); \quad i_C = C \frac{dv}{dt}(t) \]

\[ \frac{v}{R} + \frac{1}{L} \int_{t_0}^{t} v(x)dx + i_L(t_0) + C \frac{dv}{dt}(t) = i_S \]

Differentiating

\[ C \frac{d^2v}{dt^2} + \frac{1}{R} \frac{dv}{dt} + \frac{v}{L} = \frac{di_S}{dt} \]

\[ L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = \frac{dv_S}{dt} \]

Single Loop: Use KVL

\[ -v_S + v_R + v_C + v_L = 0 \]

\[ v_R = Ri; \quad v_C = \frac{1}{C} \int_{t_0}^{t} i(x)dx + v_C(t_0); \quad v_L = L \frac{di}{dt}(t) \]

\[ Ri + \frac{1}{C} \int_{t_0}^{t} i(x)dx + v_C(t_0) + L \frac{di}{dt}(t) = v_S \]
### Recall Diet Problem

<table>
<thead>
<tr>
<th></th>
<th>Grain</th>
<th>Fishmeal</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbohydrates</td>
<td>0.3</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Protein</td>
<td>0.2</td>
<td>0.8</td>
<td>1.20</td>
</tr>
<tr>
<td>Fat</td>
<td>0.1</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>Price</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
Duality Theory

minimize \[ z = 2x_1 + 4x_2 \]

subject to
\[ 0.3x_1 + 0.1x_2 \geq 0.45 \]
\[ 0.2x_1 + 0.8x_2 \geq 1.20 \]
\[ 0.1x_1 + 0.1x_2 \geq 0.3 \]
\[ x_1, x_2 \geq 0 \]
Duality Theory

Pharmaceutical company producing nutrient pills competitive with real food.

What prices should the company charge for one unit of carbohydrates, protein, fat?

<table>
<thead>
<tr>
<th>Carbohydrates</th>
<th>Protein</th>
<th>Fat</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$y_3$</td>
<td>cents/oz</td>
</tr>
</tbody>
</table>

10
maximze \quad 0.45y_1 + 1.20y_2 + 0.30y_3 \quad \Leftrightarrow \quad \text{profit, price}

subject to \quad 0.3y_1 + 0.2y_2 + 0.1y_3 \leq 2 \quad \Leftrightarrow \quad \text{grain}
\quad 0.1y_1 + 0.8y_2 + 0.1y_3 \leq 4 \quad \Leftrightarrow \quad \text{fishmeal}

\begin{align*}
\max & \quad [0.45 \quad 1.20 \quad 0.30] y \\
\text{s. t.} & \quad \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.8 & 0.1 \end{bmatrix} y \leq \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\end{align*}
Original Problem

\[
\min \begin{bmatrix} 2 & 4 \end{bmatrix} x
\]

s. t.

\[
\begin{bmatrix}
0.3 & 0.1 \\
0.2 & 0.8 \\
0.1 & 0.1
\end{bmatrix}
\begin{bmatrix} x \\
\end{bmatrix} \geq
\begin{bmatrix}
0.45 \\
1.20 \\
0.30
\end{bmatrix}
\]
Duality Theory

\[
\begin{align*}
\text{min} & \quad c^T x \\
n\text{s. t.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad b^T y \\
n\text{s. t.} & \quad A^T y \leq c \\
& \quad y \geq 0
\end{align*}
\]

\(y\): dual variable; \(m\) – dimensional vector corresponding to constraint inequalities
Linear program in standard form

\[ \text{minimize } \mathbf{c}^T \mathbf{x} \]
\[ \text{s. t. } \mathbf{A} \mathbf{x} = \mathbf{b} \]
\[ \mathbf{x} \geq \mathbf{0} \]

Equivalent form

\[ \text{minimize } \mathbf{c}^T \mathbf{x} \]
\[ \text{s. t. } \mathbf{A} \mathbf{x} \geq \mathbf{b} \]
\[ -\mathbf{A} \mathbf{x} \geq -\mathbf{b} \]
\[ \mathbf{x} \geq \mathbf{0} \]
Linear program in standard form

minimize \[ c^T x \]
subject to \[
\begin{bmatrix}
A \\
-A
\end{bmatrix}
\begin{bmatrix}
x \\
-A
\end{bmatrix}
\geq
\begin{bmatrix}
b \\
-b
\end{bmatrix}
\]
\[ \begin{array}{ccc}
\cdots & u \\
\cdots & v
\end{array} \]

Dual problem

maximize \[ b^T (u - v) \]
subject to \[ A^T (u - v) \leq c \]
\[ \begin{array}{c}
u \geq 0 \\
v \geq 0
\end{array} \]

Let \[ y = u - v : \text{ free variable} \]
Linear program in standard form

**Primal (P)**

minimize \( c^T x \)

subject to \( Ax = b \)

\( x \geq 0 \)

**Dual (D)**

maximize \( b^T y \)

subject to \( Ay \leq c \)

\( y : \text{ free} \)
Duality Theorem

**Lemma (Weak Duality Lemma):** If $x$ and $y$ are feasible for (P) and (D) respectively, then $c^T x \geq b^T y$

**Proof.** We have from (P) and (D)

$$b^T y = y^T b = y^T A x \leq c^T x$$

since $x \geq 0$ and $y^T A \leq c^T$.

**Corollary.** If $x_0$ and $y_0$ are feasible for (P) and (D), respectively, and if $c^T x_0 = b^T y_0$, then $x_0$ and $y_0$ are optimal for their respective problems.

Duality gap

$$c^T x$$

$$b^T y$$

No gap $\implies$ optimal

$$c^T x_0$$

$$b^T y_0$$
Theorem (Duality Theorem): If either of the problems (P) or (D) has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal. If either problem has an unbounded objective, the other problem has no feasible solution.

Proof. see text.

Proposition: Every \( T \) matrix looks like

\[
T = \begin{bmatrix}
1 & \pi^T \\
0 & D
\end{bmatrix}
\]

Proof. Never pivot on the objective function \(-z\)

Induction
\[
T_0 = \begin{bmatrix}
1 & 0^T \\
0 & I
\end{bmatrix}
\]
Duality Theorem of Linear Programming

Suppose

\[
T_{r-1} = \begin{bmatrix}
1 & \pi^T \\
0 & D
\end{bmatrix}
\]

\[
T_r = P_r \times T_{r-1}
= \begin{bmatrix}
1 & -\frac{c_k}{a_{hk}} \\
0 & \frac{1}{a_{hk}} \\
0 & -\frac{a_{mk}}{a_{hk}}
\end{bmatrix}
\begin{bmatrix}
1 & \pi^T \\
0 & D
\end{bmatrix}
\]
Duality Theorem of Linear Programming

\[
\begin{bmatrix}
1 & \alpha^T \\
\vdots \\
0 & \beta
\end{bmatrix}
\begin{bmatrix}
1 \\
\alpha^T \\
\vdots \\
0
\end{bmatrix}
= 
\begin{bmatrix}
1 & \pi^T \\
\pi^T + \alpha^T D \\
\vdots \\
0 & \beta D
\end{bmatrix}
= 
\begin{bmatrix}
1 & \pi^T \\
\pi^T' \\
\vdots \\
0 & D'
\end{bmatrix}
\]
Duality Theorem of Linear Programming

Let

\[
\begin{bmatrix}
1 & \hat{\pi}^T \\
0 & \hat{D}
\end{bmatrix}
\begin{bmatrix}
1 & -c^T & 0^T & 0 \\
0 & A & I & b
\end{bmatrix}
\{ 1 \}
\{ m \}
\]

\[
\begin{bmatrix}
1 & -c^T + \hat{\pi}^T A & \hat{\pi}^T & \hat{\pi}^T b \\
0 & \hat{D} A & \hat{D} & \hat{D} b
\end{bmatrix}
= \]

*Note:* Primal problem is a max problem for correct sign.
Duality Theorem of Linear Programming

**Dual m variables**

$\hat{\pi}$ is a candidate for the solution to dual

$$\hat{\pi}^T \geq 0$$  
Restriction for dual variables

$$-c^T + \hat{\pi}^T A \geq 0$$  
**Optimality condition**

$$A^T \hat{\pi} \geq c$$  
Constraints for the dual problem

$\hat{\pi}$ satisfies constraints and restrictions for dual, therefore $\hat{\pi}$ is a feasible solution to the dual problem. It is also the optimal solution.
Duality Theorem of Linear Programming

Objective function for the optimal solution is $\hat{\pi}^T b$

In other words, if $\hat{x}$ is the optimal solution to (P),

$$c^T \hat{x} = \hat{\pi}^T b$$

The value of the objective function for the dual is $b^T \hat{\pi}$
Both Primal and Dual may be infeasible.

ex) \[\max \quad 1 \cdot x\]
\[0 \cdot x \leq -1 \quad \text{infeasible}\]
\[x \geq 0\]

\[\min \quad -1 \cdot y\]
\[0 \cdot y \geq 1 \quad \text{infeasible}\]
\[y \geq 0\]
Ex. For the primal, find the dual problem.

\[
\text{max} \quad z = 25x_1 + 5x_2
\]

\[
x_1 + 2x_2 \leq 10 \quad \leftarrow y_1
\]

\[
x_1 - x_2 \leq 3 \quad \leftarrow y_2
\]

\[
3x_1 + 2x_2 \leq 14 \quad \leftarrow y_3
\]

\[
x_1 \geq 0, \ x_2 \geq 0
\]
Duality Theorem of Linear Programming

Dual

\[
\begin{align*}
\min & \quad w = 10y_1 + 3y_2 + 14y_3 \\
& \quad y_1 + y_2 + 3y_3 \geq 25 \\
& \quad 2y_1 - y_2 + 2y_3 \geq 5 \\
& \quad y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0
\end{align*}
\]

\((x_1, x_2) = (3, 2)\) \quad Any feasible solution

\[z = 85\]

\((y_1, y_2, y_3) = (4, 5, 6)\]

\[w = 139\]

Weak Duality Lemma: Always the value of the objective function of the minimization problem is bigger than that of the maximization problem

Always \(z \leq w\) \quad \Rightarrow
### Duality Theorem of Linear Programming

<table>
<thead>
<tr>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>7</th>
<th>6</th>
<th>105</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4/5</td>
<td>-3/5</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2/5</td>
<td>1/5</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-3/5</td>
<td>1/5</td>
<td>1</td>
</tr>
</tbody>
</table>
Duality Theorem of Linear Programming

\[
T = \begin{bmatrix}
1 & 0 & 7 & 6 \\
0 & 1 & 4/5 & -3/5 \\
0 & 0 & 2/5 & 1/5 \\
0 & 0 & -3/5 & 1/5
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
0 \\
7 \\
6
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 4/5 & -3/5 \\
0 & 2/5 & 1/5 \\
0 & -3/5 & 1/5
\end{bmatrix}
\]
Duality Theorem of Linear Programming

\[
\begin{aligned}
\begin{cases}
  x_1 = 4, & x_2 = 1, & z = 105 \\
  y_1 = 0, & y_2 = 7, & y_3 = 6, & w = 105 
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
  x_1 = 4 & \iff t_1 = 0 \\
  x_2 = 1 & \iff t_2 = 0 \\
  s_1 = 4 & \iff y_1 = 0 \\
  s_2 = 0 & \iff y_2 = 7 \\
  s_3 = 0 & \iff y_3 = 6 \\
\end{aligned}
\]

Eureka!

**Claim:** For optimal solutions for the primal and dual problems, always one of the pair is zero.
Duality Theorem of Linear Programming

**Claim:** For optimal solutions for the primal and dual problems, always one of the pair is zero

⇒ **Complementary Slackness Condition (C. S. C.)**

\[
\begin{align*}
\text{max} & \quad c^T x \\
Ax & \leq b \\
x & \geq 0 \\
\text{min} & \quad b^T y \\
A^T y & \geq c \\
y & \geq 0
\end{align*}
\]

Note:
Primal problem is a max problem for correct sign
Duality Theorem of Linear Programming

Proof. \[ c^T x \leq y^T A x = x^T A^T y \leq b^T y \]

whenever \( x \) and \( y \) are feasible in P and D.

Suppose \( \hat{x} \) and \( \hat{y} \) are optimal for P and D, then, from WDL we have

\[ c^T \hat{x} \leq \hat{y}^T A \hat{x} = \hat{x}^T A^T \hat{y} \leq b^T \hat{y} = c^T \hat{x} \] \( \iff \) \text{Optimality}

becomes

\[ c^T \hat{x} = \hat{y}^T A \hat{x} = \hat{x}^T A^T \hat{y} = b^T \hat{y} \]

\[ \hat{x}^T c = \hat{x}^T A^T \hat{y} \]

\[ \hat{y}^T A \hat{x} = \hat{y}^T b \]

\[ \hat{x}^T (A^T \hat{y} - c) = 0 \]

\[ \hat{y}^T (b - A \hat{x}) = 0 \]
Two complementary slackness conditions

\[ \hat{x}^T (A^T \hat{y} - c) = 0 \]

\[ A^T \hat{y} - \hat{t} = c \quad \hat{t} \geq 0 \quad \text{Surplus variable} \]

\[ A^T \hat{y} - c = \hat{t} \]

\[ \hat{x}^T \hat{t} = 0 \]

\[ \sum_{i=1}^{n} \hat{x}_i \cdot \hat{t}_i = 0 \quad \hat{x}_i, \hat{t}_i \geq 0 \]

\[ \hat{x}_i \cdot \hat{t}_i = 0, \quad \text{for } i = 1, 2, \ldots, n \]
Two complementary slackness conditions

From
\[ \hat{y}^T (b - A\hat{x}) = 0 \]
\[ A\hat{x} + \hat{s} = b \]
\[ \hat{y}^T \hat{s} = 0 \]
\[ \sum_{i=1}^{m} \hat{y}_i \cdot \hat{s}_i = 0, \quad \hat{y}_i, \hat{s}_i \geq 0 \]
\[ \hat{y}_i \cdot \hat{s}_i = 0 \quad \text{for } i = 1, 2, \ldots, m \]
Complementary slackness theorem

Theorem Complementary slackness theorem

A pair of feasible solution $\hat{x}$ and $\hat{y}$ to P and D, respectively are optimal, if and only if the complementary slackness conditions hold.

Proof Omitted.
Dual transform from general primal

\[
\begin{align*}
\text{max} & \quad c^T x \\
Ax & \leq b \\
x & \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad b^T y \\
A^T y & \geq c \\
y & \geq 0
\end{align*}
\]

Example

\[
\begin{align*}
\text{min} & \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 & \leq b_1 \\
a_{21} x_1 + a_{22} x_2 + a_{23} x_3 & = b_2 \\
a_{31} x_1 + a_{32} x_2 + a_{33} x_3 & \geq b_3 \\
x_1 & \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ is free}
\end{align*}
\]
Dual transform from general primal

\[
\begin{align*}
\text{min} & \quad c_1 x_1 - c_2 x_2' + c_3 x_3' - c_3 x_3'' \\
& a_{11} x_1 - a_{12} x_2' + a_{13} x_3' - a_{13} x_3'' \leq b_1 \\
& a_{21} x_1 - a_{22} x_2' + a_{23} x_3' - a_{23} x_3'' = b_2 \\
& a_{31} x_1 - a_{32} x_2' + a_{33} x_3' - a_{33} x_3'' \geq b_3 \\
& x_1 \geq 0, \quad x_2' \geq 0, \quad x_3' \geq 0, \quad x_3'' \geq 0 \\
\end{align*}
\]

\[a = b \quad \iff \quad a \leq b \quad \text{and} \quad a \geq b\]

\[
\begin{align*}
& a_{21} x_1 - a_{22} x_2' + a_{23} x_3' - a_{23} x_3'' \leq b_2 \\
& -a_{21} x_1 + a_{22} x_2' - a_{23} x_3' + a_{23} x_3'' \leq -b_2
\end{align*}
\]
Dual transform from general primal

\[
\begin{align*}
\text{max} & \quad -c_1 x_1 + c_2 x_2' - c_3 x_3' + c_3 x_3'' \\
a_{11} x_1 - a_{12} x_2' + a_{13} x_3' - a_{13} x_3'' & \leq b_1 \quad z_1 \\
a_{21} x_1 - a_{22} x_2' + a_{23} x_3' - a_{23} x_3'' & \leq b_2 \quad z_2 \\
-a_{21} x_1 + a_{22} x_2' - a_{23} x_3' + a_{23} x_3'' & \leq -b_2 \quad z_3 \\
-a_{31} x_1 + a_{32} x_2' - a_{33} x_3' + a_{33} x_3'' & \leq -b_3 \quad z_4 \\
x_1 \geq 0, x_2' \geq 0, x_3' \geq 0, x_3'' \geq 0
\end{align*}
\]
Dual transform from general primal

**Dual problem**

\[
\begin{align*}
\min \quad & b_1 z_1 + b_2 z_2 - b_3 z_3 - b_4 z_4 \\
& a_{11} z_1 + a_{21} z_2 - a_{21} z_3 - a_{31} z_4 \geq -c_1 \\
& -a_{12} z_1 - a_{22} z_2 + a_{22} z_3 + a_{32} z_4 \geq c_2 \\
& a_{13} z_1 + a_{23} z_2 - a_{23} z_3 - a_{33} z_4 \geq -c_3 \\
& -a_{13} z_1 - a_{23} z_2 + a_{23} z_3 + a_{33} z_4 \geq c_3 \\
& z_1, z_2, z_3, z_4 \geq 0 \\
y_2 = z_3 - z_2
\end{align*}
\]
Dual transform from general primal

\[ \begin{align*} 
\min & \quad b_1z_1 - b_2y - b_3z_4 \\
- & \quad a_{11}z_1 + a_{21}y + a_{31}z_4 \leq c_1 \\
- & \quad a_{12}z_1 + a_{22}y + a_{32}z_4 \geq c_2 \\
- & \quad a_{13}z_1 + a_{23}y + a_{33}z_4 = c_3 \\
\end{align*} \]

\[ z_1 \geq 0, \quad y: \text{ free}, \quad z_4 \geq 0 \]
Dual transform from general primal

Let \( y_1 = -z_1, \ y_1 \leq 0 \)

\[
\begin{align*}
\text{max} & \quad b_1 y_1 + b_2 y + b_3 z_4 \\
& \quad a_{11} y_1 + a_{21} y + a_{31} z_4 \leq c_1 \\
& \quad a_{12} y_1 + a_{22} y + a_{32} z_4 \geq c_2 \\
& \quad a_{13} y_1 + a_{23} y + a_{33} z_4 = c_2
\end{align*}
\]
Dual transform from general primal

Let \( y_3 = z_4, \ y_3 \geq 0, \ y_2 = y \): free

\[
\begin{align*}
\text{max} \quad & b_1 y_1 + b_2 y_2 + b_3 y_3 \\
& a_{11} y_1 + a_{21} y_2 + a_{31} y_3 \leq c_1 \\
& a_{12} y_1 + a_{22} y_2 + a_{32} y_3 \geq c_2 \\
& a_{13} y_1 + a_{23} y_2 + a_{33} y_3 = c_2 \\
& y_1 \leq 0, \ y_2 : \text{free}, \ y_3 \geq 0
\end{align*}
\]
## Dual transform from general primal

<table>
<thead>
<tr>
<th>General Dual</th>
<th>General Primal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leq ) constraint</td>
<td>( \leq 0 ) variable</td>
</tr>
<tr>
<td>= constraint</td>
<td>free variable</td>
</tr>
<tr>
<td>( \geq ) constraint</td>
<td>( \geq 0 ) variable</td>
</tr>
<tr>
<td>( \geq 0 ) variable</td>
<td>( \leq ) constraint</td>
</tr>
<tr>
<td>( \leq 0 ) variable</td>
<td>( \geq ) constraint</td>
</tr>
<tr>
<td>free variable</td>
<td>= constraint</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\min & \quad b^T y \\
A^T y & \geq c \\
y & \geq 0
\end{align*}
\quad \quad \quad \quad \quad \quad
\begin{align*}
\max & \quad c^T x \\
A x & \leq b \\
x & \geq 0
\end{align*}
\]
Dual transform from general primal

Example

\[
\begin{align*}
\text{max} & \quad x_1 - 2x_2 \\
x_1 - x_2 & \leq 5 & y_1 \\
3x_1 - x_2 & \geq 7 & y_2 \\
-x_1 + x_2 & \leq -1 & y_3 \\
x_1 & \geq 0, & x_2 & \leq 0
\end{align*}
\]
Dual transform from general primal

\[
\begin{align*}
\text{min} & \quad 5y_1 + 7y_2 - y_3 \\
y_1 + 3y_2 - y_3 & \geq 1 \\
-y_1 - y_2 + y_3 & \leq -2 \\
y_1 & \geq 0, \quad y_2 \leq 0, \quad y_3 \geq 0
\end{align*}
\]
**Dual transform from general primal**

Simplex tableau for the example (Primal problem is a max problem)

\[
\begin{array}{ccccccc}
1 & -1 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 5 \\
0 & 3 & 1 & 0 & -1 & 0 & 7 \\
0 & 1 & 1 & 0 & 0 & -1 & 1 \\
\end{array}
\]
Dual transform from general primal

Final tableau

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
</tbody>
</table>

$y_1 \begin{array}{c} 7 \\ 4 \\ 2 \\ 3 \end{array}$

$x_1 = 3, \quad x_2 = -2$
Dual transform from general primal

\[ \mathbf{T} = \begin{bmatrix} 1 & 3/2 & 0 & 1/2 \\ 0 & 2 & 1 & -1 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & -1/2 \end{bmatrix} \]

\[ y_1 = 3/2, \quad y_2 = 0, \quad y_3 = 1/2 \]
Dual transform from general primal

\[ P : \begin{align*}
\text{min} & \quad -x_1 + 2x_2 \\
& \quad x_1 - x_2 \leq 5 \\
& \quad 3x_1 - x_2 \geq 7 \\
& \quad -x_1 + x_2 \leq -1 \\
& \quad x_1 \geq 0, \ x_2 \leq 0
\end{align*} \]

\[ D : \begin{align*}
\text{max} & \quad 5y_1 + 7y_2 - y_3 \\
& \quad y_1 + 3y_2 - y_3 \leq -1 \\
& \quad -y_1 - y_2 + y_3 \geq 2 \\
& \quad y_1 \leq 0, \ y_2 \geq 0, \ y_3 \leq 0
\end{align*} \]
Dual transform from general primal

1) \( \min P \Rightarrow \max D \)
   \[ y_1 = -\frac{3}{2}, \quad y_2 = 0, \quad y_3 = -\frac{1}{2} \]
   \( \Rightarrow \) Dual solution: negative of \( \pi \)

2) \( \max P \Rightarrow \min D \)
   \( \Rightarrow \) Dual solution: \( \pi \) correct sign
Dual transform from general primal

\[
\begin{align*}
\text{max} & \quad 25x_1 + 5x_2 \\
& \quad x_1 + 2x_2 \leq 10 \quad y_1 \\
& \quad x_1 - x_2 \leq 3 \quad y_2 \\
& \quad 3x_1 + 2x_2 \leq 14 \quad y_3 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad 10y_1 + 3y_2 + 14y_3 \\
& \quad y_1 + y_2 + 3y_3 \geq 25 \\
& \quad 2y_1 - y_2 + 2y_3 \geq 5 \\
& \quad y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0
\end{align*}
\]
## Dual transform from general primal

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>7</th>
<th>6</th>
<th>105</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4/5</td>
<td>-3/5</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2/5</td>
<td>1/5</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-3/5</td>
<td>1/5</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
x_1 = 4 \\
x_2 = 1 \\
s_1 = 4 \\
s_2 = 0 \\
s_3 = 0 \\
z = 105
\]

\[
t_1 = 0 \\
t_1 = 0 \\
y_1 = 0 \\
y_2 = 7 \\
y_3 = 6 \\
w = 105
\]

Complementary Slackness Condition