Abstract—A new scheme is proposed to optimally reduce the modulation modes for bit and power allocation algorithm in multi-carrier transmissions. By reducing the modulation modes, computational load and hardware complexity can be reduced without any performance degradation. When modulation modes for rectangular QAM are available at a transmitter, it is shown that the modulation modes for rectangular QAM can be reduced to those for square QAM which is the subset of rectangular QAM. After identifying a power allocation property of rectangular QAM, it is proved that the optimal allocation with square QAM has exactly the same performance as that with rectangular QAM. Numerical results show the reduction of computational load and verify our theoretic presentations.

I. INTRODUCTION

With high-speed wireless services increasingly in demand, there is a need for more throughput per bandwidth to accommodate more users with higher data rates while retaining a guaranteed quality of service. To improve the spectral efficiency of multi-carrier transmissions, bits and power can be allocated to subcarriers according to their channel qualities which are assumed to be known at a transmitter [1]–[3]. In particular, the subcarriers with large channel gains employ higher order modulations to carry more bits per a symbol, while the subcarriers with low channel gains carry one or even zero bit.

Many bit and power allocation algorithms for multi-carrier transmission schemes have been proposed. Hughes-Hartogs algorithm [1] can be applied to minimize the total transmission power with the constraint of the fixed channel capacity or to maximize the channel capacity with the total transmission power constraint. Chow’s algorithm [2] is a computationally efficient bit and power allocation algorithm for maximizing the channel capacity. Krongold [3] used a Lagrange multiplier bisection search to find the optimal bit and power allocation effectively constrained by the constant transmission power and a required bit error rate (BER). For these allocation algorithms, the quadrature amplitude modulation (QAM) with Gray bit mapping is commonly used because of its inherent spectral efficiency and the ease of implementation [4]. Rectangular QAM is used in [2], [3], [5] and square QAM, which is the efficient subset of rectangular QAM, is adopted in [6], [7].

In this letter, we formulate a bit and power allocation problem for multi-carrier transmission schemes to a constrained optimization problem, in which overall transmission power is minimized with the constraints of a fixed data rate and BER. Rectangular QAM and square QAM are investigated as available modulation schemes. In this allocation problem, it is proved that the optimal allocation with square QAM has exactly the same overall transmission power as that with rectangular QAM. From this result, it is also proved that the optimal allocation with square QAM is also optimal to the allocation problem with rectangular QAM. Therefore the available modulation modes can be reduced to those for square QAMs, and thus computational load and hardware complexity can be reduced without any performance degradation.

II. PROBLEM FORMULATION

We consider an adaptive multi-carrier transmission system in which the transmitter uses combined bit and power allocation algorithms based on the channel information. A modulation mode for each subcarrier is selected corresponding to the number of bits allocated to the subcarrier and the symbol modulated by the selected mode is then scaled to the allocated power. We define \(c(k)\) and \(p(k)\) as the number of allocated bits and the transmission power level of the \(k\)th subcarrier, respectively.

Assuming that the noise of each subcarrier has the same power, we denote \(f(c(k))\) as the required received power in the \(k\)th subcarrier to satisfy a given BER requirement in a \(c(k)\) bits/symbol modulation scheme. \(f(c(k))\) can be obtained by using the error functions of the corresponding modulation schemes and the noise power [4]. Assuming that the channel state information is known, the transmission power of the \(k\)th subcarrier can be given by

\[
p(k) = f(c(k))/|H(k)|^2,
\]
where $H(k)$ is the channel gain of the $k$th subcarrier. Thus the combined bit and power allocation algorithm should find the optimal assignment of $c(k)$ so that the total transmission power is minimized satisfying the transmission rate and BER requirements. The optimization problem can be mathematically formulated as following:

$$
\min_{c(k) \in D} P_T = \sum_{k=1}^{K} f(c(k))/|H(k)|^2
$$

subject to

$$
\sum_{k=1}^{K} c(k) = R_T,
$$

where $D$ is the set of all possible values for $c(k)$, $K$ is the number of subcarriers, and $c(k) = 0$ means that no information is transmitted through the $k$th subcarrier.

III. M-ARY RECTANGULAR QAM

Rectangular QAM is considered as available modulation scheme of (2). Rectangular QAM can be implemented with two independent pulse amplitude modulations (PAM); one for the in-phase branch, and the other for the quadrature-phase branch [8]. The modulated signal of the $k$th subcarrier is expressed as

$$
s_k(n) = A_{Ik} \cos(2\pi f_k n) - A_{Qk} \sin(2\pi f_k n),
$$

where $A_{Ik}$ and $A_{Qk}$ are the signal amplitudes of in-phase and quadrature-phase, respectively, $f_k$ is the frequency of the $k$th subcarrier and $N$ is the symbol interval.

In M-ary rectangular QAM, $c \triangleq \log_2 M$ bits are divided into halves, that is, $c_I$ bits and $c_Q$ bits. They are mapped onto in-phase and quadrature-phase respectively. If $c$ is even, $c$ is split into halves, that is, $c_I = c_Q = c/2$. However, if $c$ is odd, $c$ can not be split into the same integers. Thus $c_I$ and $c_Q$ are defined as

$$
c_I \triangleq \lfloor c/2 \rfloor,
c_Q \triangleq \lceil c/2 \rceil,
$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor function and the ceiling function, respectively. Table I shows $c_I$ and $c_Q$ according to $c$.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_I$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>...</td>
</tr>
<tr>
<td>$c_Q$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>...</td>
</tr>
</tbody>
</table>

Using Gray coding, $c_I$ bits are mapped onto the in-phase channel whose amplitude $A_{Ik}$ is selected over the set of $\{\pm d, \pm 3d, \cdots, \pm(2^{c_I}-1)d\}$. Similarly, $c_Q$ bits are mapped onto the quadrature-phase channel, whose amplitude $A_{Qk}$ is selected over the set of $\{\pm d, \pm 3d, \cdots, \pm(2^{c_Q}-1)d\}$.

To explain the power relationship between PAM and QAM, let us denote the required received power, $f(\cdot)$ for PAM and QAM as $f_p(\cdot)$ and $f_q(\cdot)$. Then, $f_p(c)$ and $f_q(c)$ denote the required received power for PAM and QAM to transmit $c$ bits per a symbol with satisfying a given BER requirement.

Because rectangular QAM can be represented with two PAMs, the power of rectangular QAM is the sum of the power of two PAMs as

$$
f_q(c) = f_p(c_I) + f_p(c_Q).
$$

Specifically, for odd $c$’s, (4) can be represented as

$$
f_q(c) = f_p(c + 1) + f_p(c - 1),
$$

and for even $c$’s,

$$
f_q(c) = f_p(c/2) + f_p(c/2).
$$

Combining (5) and (6), we obtain the following property.

Property 1 For odd $c$,

$$
f_q(c + 1) = f_q(c) = f_q(c) - f_q(c - 1).
$$

Property 1 means that the power increments to transmit one more bit per a symbol are the same for $c$ bits/symbol and $(c - 1)$ bits/symbol modulation modes, if $c$ is odd.

IV. REDUCTION OF MODULATION MODES

In the bit and power allocation problem (2), $D$ is the set of the possible numbers for $c(k)$. To compare rectangular QAM and square QAM, the possible number set of rectangular QAM is denoted by $D_1$, and that of square QAM by $D_2$, where $D_1 \triangleq \{0, 1, 2, 3, \cdots\}$ and $D_2 \triangleq \{0, 2, 4, 6, \cdots\}$. Assuming that the overall transmission rate, $R_T$, in the allocation problem (2) is even, the following lemma can be obtained based on Property 1.

Theorem 1 A bit and power allocation with rectangular QAM has the same minimum transmission power as that with square QAM when overall rate requirement, $R_T$, is even.

$\textbf{Proof:}$ First, consider the bit and power allocation problem with rectangular QAM. An optimal solution, $\{c_I(k), \forall k\}$, of the optimization problem (2) with $D = D_1$ exists and can be obtained by the greedy approach [1].

The number of odd elements in the optimal solution, $\{c_I(k), \forall k\}$, is even because $\sum_{k=1}^{K} c_I(k) = R_T$ and $R_T$ is even. If odd elements exist in the optimal solution, the odd elements can be paired into $(c_I^o(i), c_I^o(j))$ so that the following inequality holds:

$$
\frac{f(c_I^o(i)) - f(c_I^o(i) - 1)}{|H(i)|^2} \leq \frac{f(c_I^o(j)) - f(c_I^o(j) - 1)}{|H(j)|^2},
$$

where $i$ and $j$ are subcarrier indices and $i \neq j$.

As $c_I^o(i)$ is odd, Property 1 gives the following equation:

$$
f(c_I^o(i) + 1) - f(c_I^o(i)) = f(c_I^o(i)) - f(c_I^o(i) - 1).
$$
From (8) and (9), we can obtain
\[
\frac{f(c_i^*(i) + 1) - f(c_i^*(i))}{|H(i)|^2} \leq \frac{f(c_j^*(j)) - f(c_j^*(j) - 1)}{|H(j)|^2},
\]
(10)

or
\[
\frac{f(c_i^*(i) + 1)}{|H(i)|^2} + \frac{f(c_j^*(j) - 1)}{|H(j)|^2} \leq \frac{f(c_i^*(i))} {|H(i)|^2} + \frac{f(c_j^*(j))}{|H(j)|^2}.
\]
(11)

In order to replace the pair of odd numbers, \((c_i^*(i), c_j^*(j))\), by the pair of even numbers, \((c_i^*(i) + 1, c_j^*(j) - 1)\), a new solution, \(\{c'_i(k), \forall k\}\), is defined as
\[
\begin{align*}
c'_i(k) &= c_i^*(k) + 1, \quad k = i, \\
&= c_i^*(k) - 1, \quad k = j, \\
&= c_i^*(k), \quad \text{otherwise}.
\end{align*}
\]
(12)

The optimality of the solution, \(\{c'_i(k), \forall k\}\), can be checked by the rate constraint and the minimum transmission power. By the definition (12), the solution, \(\{c'_i(k), \forall k\}\), satisfies the rate constraint as following:
\[
\sum_{k=1}^{K} c'_i(k) = \sum_{k=1}^{K} c_i^*(k) = R_T.
\]
(13)

(11) can be rewritten with \(\{c'_i(k), \forall k\}\) as
\[
\frac{f(c'_i(k))}{|H(i)|^2} + \frac{f(c'_j(k))}{|H(j)|^2} \leq \frac{f(c_i^*(i))}{|H(i)|^2} + \frac{f(c_j^*(j))}{|H(j)|^2}.
\]
(14)

and thus
\[
\sum_{k=1}^{K} \frac{f(c'_i(k))}{|H(i)|^2} \leq \sum_{k=1}^{K} \frac{f(c_i^*(k))}{|H(i)|^2}.
\]
(15)

On the other side, because the optimal solution, \(\{c_i^*(k), \forall k\}\), has the minimum cost, the following inequality holds:
\[
\sum_{k=1}^{K} \frac{f(c_i^*(k))}{|H(i)|^2} \geq \sum_{k=1}^{K} \frac{f(c'_i(k))}{|H(i)|^2}.
\]
(16)

From (15) and (16),
\[
\sum_{k=1}^{K} \frac{f(c'_i(k))}{|H(i)|^2} = \sum_{k=1}^{K} \frac{f(c_i^*(k))}{|H(i)|^2}.
\]
(17)

By (13) and (17), \(\{c'_i(k), \forall k\}\) is optimal to the allocation problem (2) with \(D = D_1\).

In the same way, all the remaining pairs of odd numbers in the optimal solution, \(\{c_i^*(k), \forall k\}\), can be replaced by the pairs of even numbers maintaining the optimality. Therefore there always exists an optimal solution which is composed of only even numbers. This optimal solution is denoted by \(\{c^*(k), \forall k\}\), and \(\{c^*(k), \forall k\} \in D_2\).

The minimum transmission power with \(D = D_2\) can not be less than that with \(D = D_1\) because \(D_2 \subset D_1\). Therefore \(\{c^*(k), \forall k\}\) is the optimal solution of the allocation problem (2) with \(D = D_2\).

Consequently there always exists a common optimal solution of the allocation problems (2) with \(D = D_1\) and \(D = D_2\), and thus the bit and power allocations with \(D = D_1\) and \(D = D_2\) provide the same minimum transmission power when \(R_T\) is even.

It was proved that there always exists a common optimal solution of two allocation problems, that is, the allocation problems with square QAM and rectangular QAM. Furthermore, in the next lemma, it will be proved that all the optimal solutions of the allocation problem with square QAM are those common optimal solutions.

**Theorem 2** All the optimal solutions of a bit and power allocation problem with square QAM are optimal to that with rectangular QAM when overall rate requirement, \(R_T\), is even.

**Proof:** Let \(\{c^*_2(k), \forall k\}\) be any optimal solution of the allocation problem (2) with \(D = D_2\). \(\{c^*_2(k), \forall k\}\) also achieves the minimum transmission power of the allocation problem (2) with \(D = D_1\) by Lemma 1, and \(\{c^*_2(k), \forall k\} \in D_1\) because \(D_2 \subset D_1\). Therefore all the optimal solutions of the allocation problem (2) with \(D = D_2\) are optimal to that with \(D = D_1\).

As a result, two allocation problems with rectangular QAM and square QAM have the same result, and the optimal allocation with rectangular QAM can be provided by solving the allocation problem with square QAM. Therefore the optimal allocation can be searched over the modulation modes for only square QAM, and it is unnecessary to use all modulation modes for rectangular QAM. Because available modulation modes are reduced to a half, we can reduce the computational load to solve the allocation problem, and also reduce the hardware complexity to modulate bits by the corresponding modulation mode.

**V. Numerical Results**

In order to verify our theoretical results and compare the computational complexities of the bit and power allocations with rectangular QAM and square QAM, simulations were performed in the conditions of an adaptive, uncoded OFDM system with 64 subcarriers, using 1,000 sets of fading channels generated by HyperLAN/2 channel model B. Target BER is set to \(10^{-3}\). The simulations are carried out with \(10 \leq R_T \leq 200\).

Figure 1 shows the transmit signal to noise ratio (SNR), which is the total transmission power divided by noise power. This result shows that the bit and power allocation with rectangular QAM and square QAM have exactly the same transmission power and therefore our theoretical results are confirmed.

Figure 2 depicts the computational loads to solve the bit and power allocation problem. We employ the greedy approach in [1] to obtain the optimal solutions. The algorithms were coded in MATLAB and the platform was an intel pentium-4 2.4 GHz personal computer. The bit and power allocation with square
QAM is twice faster than that with rectangular QAM because the number of available modulation modes is reduced to a half.

VI. CONCLUSION

We presented that the optimal solution of the bit and power allocation problem can be searched over only square QAM without any performance degradation, where the computational load is reduced to a half. For this purpose, we first identified the power allocation property of rectangular QAM. Based on this property, we showed that the performance of the bit and power allocation algorithm does not degrade although the modulation modes are reduced to those for square QAM. The simulation results showed the validity of theoretical results and the reduction of computational load.

REFERENCES